r is the latent of vaporization;
t is the time;
X is the point of the plane;
x_1 is the longitudinal coordinate;
x_2 is the transverse coordinate;
Δt is the step in time;
H is the step along x_1;
H2 is the step along x_2;
ε is the parameter of difference scheme.

Indices

0 is the initial;
s is the boundary;
m is the number for time point;
w is the number for coolant parameters;
h is the number for grid function.

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METHOD OF EXTENSION OF THE DOMAIN OF HEAT- AND MOISTURE-CONDUCTION PROBLEMS

I. Z. Aktuganov

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A method based on extension of the domain of the problem is applied to the solution of parabolic differential equations in heat- and moisture-conduction problems.

There is a well-known method for the solution of elasticity problems by extension of the domain of definition [1, 2]. A similar approach is possible in heat- and moisture-conduction problems for the solution of differential equations of parabolic type.

Let it be required to determine a function T(r, τ) continuous and defined in a closed domain D, in which it satisfies the differential equation

$$\frac{\partial T(r,\ \tau)}{\partial \tau} = a\nabla^2 T(r,\ \tau) + \varphi(r,\ \tau), \tag{1}$$

the initial condition

$$v(r) = T(r,\ 0), \tag{2}$$

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and the following boundary conditions on the surface S bounding D:

$$\alpha \frac{\partial T(r, \tau)}{\partial n}\Big|_S + \beta T(r, \tau)\Big|_S = \gamma\psi(P, \tau), \tag{3}$$

where r is the position vector of a point with coordinates $\{x, y, z\}$; a is a parameter characterizing the properties of the domain D; and α, β, γ, are parameters used for variation of the boundary conditions; in particular for heat-conduction problems condition (3) goes over to a Dirichlet boundary condition for the set of parameters ($\alpha = 0$, $\beta = 1$, $\gamma = \gamma_1$, $\psi = \psi_1$), to a Neumann condition for ($\alpha = \alpha_2$, $\beta = 0$, $\gamma = \gamma_2$, $\psi = \psi_2$), and to a Cauchy condition for ($\alpha = \alpha_3$, $\beta = \beta_3$, $\gamma = \gamma_3$, $\psi = \psi_3$).

In place of the original domain D we consider unbounded space and delineate in it the domain B bounded by a surface Ω congruent with the surface S of the original domain D. We also identify a closed surface F exterior to the domain B.

We seek a function $\theta(r, \tau)$ defined in unbounded space and satisfying the differential equation

$$\frac{\partial\theta(r, \tau)}{\partial\tau} = a\nabla^2\theta(r, \tau) + \varphi(r, \tau) + q(r, \tau) \tag{4}$$

and the initial condition

$$v(r) = \theta(r, 0). \tag{5}$$

We impose the following constraints on the functions $\varphi(r, \tau)$, $q(r, \tau)$, and $v(r)$:

1) The functions $\varphi(r, \tau)$ and $v(r)$ have the same values in B as in problem (1)-(3) and are equal to zero outside B.

2) The function $q(r, \tau)$ is nonvanishing only on the surface F.

We invoke the Green's function $G(r - r_0, \tau - t)$, which is the solution of the equation

$$\frac{\partial G(r - r_0, \tau - t)}{\partial\tau} = a\nabla^2 G(r - r_0, \tau - t) + \delta(r - r_0, \tau - t) \tag{6}$$

for unbounded space with a homogeneous initial condition. In Eq. (6) $\delta(r - r_0, \tau - t)$ is the Dirac delta function.

Now the solution of problem (4)-(5) can be written in the form

$$\theta(r, \tau) = \int_0^\tau \Big[\iint_F q(r_0, t)\, G(r - r_0, \tau - t)\, dF\Big] dt + \int_0^\tau \Big[\iiint_D \varphi(r_0, t)\, G(r - r_0, \tau - t)\, dV\Big] dt + \iiint_D v(r_0)\, G(r - r_0, \tau)\, dV. \tag{7}$$

We determine the unknown function $q(r_0, t)$ on the basis of satisfying the boundary condition (3) on the surface Ω of unbounded space:

$$\alpha \frac{\partial\theta(r, \tau)}{\partial n}\Big|_\Omega + \beta\theta(r, \tau)\Big|_\Omega = \gamma\psi(P, \tau). \tag{8}$$

Determining the value of the function $\theta(r, \tau)$ from expression (7) for the appropriate values of the vector r, we substitute it into Eq.(8) along with the gradient

$$\frac{\partial\theta(r, \tau)}{\partial n} = \int_0^\tau \Big[\iint_F q(r_0, t) \frac{\partial G(r - r_0, \tau - t)}{\partial n} dF\Big] dt + \int_0^\tau \Big[\iiint_D \varphi(r_0, t) \frac{\partial G(r - r_0, \tau - t)}{\partial n} dV\Big] dt + \iiint_D v(r_0) \frac{\partial G(r - r_0, \tau)}{\partial n} dV. \tag{9}$$

As a result, we obtain an integral equation for a point of the surface Ω with respect to the function $q(r_0, t)$:

$$\int_0^\tau \left\{ \iint_F q(r_0, t) \left[\beta G(r-r_0, \tau-t) + \alpha \frac{\partial G(r-r_0, \tau-t)}{\partial n} \right] dF \right\} dt = H(r, \tau), \tag{10}$$

where

$$H(r, \tau) = \gamma\psi(P, \tau) - \int_0^\tau \left\{ \iiint_D \varphi(r_0, t) \left[\beta G(r-r_0, \tau-t) + \alpha \frac{\partial G(r-r_0, \tau-t)}{\partial n} \right] dV \right\} dt - \iiint_D v(r_0) \left[\beta G(r-r_0, \tau) + \alpha \frac{\partial G(r-r_0, \tau)}{\partial n} \right] dV. \tag{11}$$

The application of condition (10) to the function $q(r, \tau)$ ensures identity of the solutions $T(r, \tau)$ and $\theta(r, \tau)$ in the congruent domains D and B:

$$T(r, \tau) \equiv \theta(r, \tau). \tag{12}$$

Thus, in both problems the initial conditions, the function $\varphi(r, \tau)$ in the interiors of D and B, and the boundary conditions on their surfaces S and Ω are identical, and the function $q(r, \tau)$ vanishes in the interior of B.

Equation (10) is a Fredholm integral equation in the variable r and a Volterra integral equation in the time τ. Consequently, the problem of finding a solution to the differential equation (3) in a closed domain reduces to the problem of finding a solution of the integral equation (10) for the function $q(r, \tau)$. In this respect the given method is similar to the method of thermodynamic potentials.

We now discuss one of the possible means of solving Eq. (10); the method relies on putting the problem in discrete form. We overlay the surface Ω with a grid having a definite mesh configuration. We associate with every node i ($i = \overline{1, n}$) of the grid with coordinates r_i on Ω a node j ($j = \overline{1, n}$) with coordinate r_j on F. In place of the continuous function $q(r, \tau)$ we consider a discrete function $Q(r_j, \tau)$, which is nonvanishing only at the n nodal points of the surface F. We define it as the integral value of the function $q(r, \tau)$ over an element f_j of F:

$$Q(r_j, \tau) = \iint_{f_j} q(r, \tau)\, dF. \tag{13}$$

We require satisfaction of the boundary condition (8) at a finite number n of nodes of the surface $\Omega \equiv S$. As a result, we obtain a system of Volterra integral equations for the determination of $Q(r_j, \tau)$ ($j = \overline{1, n}$), writing it in the matrix form

$$\int_0^\tau [L(\tau-t)]\{Q(t)\}\, dt = \{H(\tau)\}, \tag{14}$$

where

$$\{Q(t)\}^T = \{Q(r_1, t),\ Q(r_2, t),\ \ldots,\ Q(r_j, t),\ \ldots,\ Q(r_n, t)\}; \tag{15}$$

$$\{H(\tau)\}^T = \{H(r_1, \tau),\ H(r_2, \tau),\ \ldots,\ H(r_i, \tau),\ \ldots,\ H(r_n, \tau)\}; \tag{16}$$

$$[L(\tau-t)] = \begin{bmatrix} L_{11}(\tau-t) & L_{12}(\tau-t) & \ldots & L_{1j}(\tau-t) & \ldots & L_{1n}(\tau-t) \\ L_{21}(\tau-t) & L_{22}(\tau-t) & \ldots & L_{2j}(\tau-t) & \ldots & L_{2n}(\tau-t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{i1}(\tau-t) & L_{i2}(\tau-t) & \ldots & L_{ij}(\tau-t) & \ldots & L_{in}(\tau-t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{n1}(\tau-t) & L_{n2}(\tau-t) & \ldots & L_{nj}(\tau-t) & \ldots & L_{nn}(\tau-t) \end{bmatrix}. \tag{17}$$

The elements of the matrix (17) are specified by the relations

$$L_{ij}(\tau-t) = \beta_i G(r_i - r_j, \tau-t) + \alpha_i \frac{\partial G(r_i - r_j, \tau-t)}{\partial n_i} \quad (i, j = \overline{1, n}). \tag{18}$$

The elements of the vector $\{H(\tau)\}$ are given by expression (11), to which an appropriate quadrature formula is applicable, depending on the form of the functions $\varphi(r, \tau)$ and $v(r)$. In particular, we can use the discretization procedure described above to partition the domain D into l elements of volume d_p ($p = \overline{1, l}$) in such a way that

$$D = \sum_{p=1}^{l} d_p. \tag{19}$$

We specify the coordinates of the centers of the elements d_p as follows:

$$r_{pv} = \iiint_{d_p} v(r)\, r\, dV \Big/ \iiint_{d_p} v(r)\, dV; \tag{20}$$

$$r_{p\varphi} = \iiint_{d_p} \varphi(r, \tau)\, r\, dV \Big/ \iiint_{d_p} \varphi(r, \tau)\, dV. \tag{21}$$

In the special case $v(r) = v$, $\varphi(r, \tau) = \varphi(\tau)$ the centers of the elements coincide with the geometric centers:

$$r_p = r_{pv} = r_{p\varphi}. \tag{22}$$

In place of the distributed functions $v(r)$ and $\varphi(r, \tau)$ we consider the discrete functions $V(r_p)$ and $\Phi(r_p, \tau)$, which are defined as the integral values of the functions $v(r)$ and $\varphi(r, \tau)$ over the individual elements d_p and are applied at the points r_{pv} and $r_{p\varphi}$:

$$V(r_{pv}) = \iiint_{d_p} v(r)\, dV \quad (p = \overline{1, l}); \tag{23}$$

$$\Phi(r_{p\varphi}, \tau) = \iiint_{d_p} \varphi(r, \tau)\, dV \quad (p = \overline{1, l}). \tag{24}$$

Then

$$H(r_i, \tau) = \gamma\psi(P_i, \tau) - \sum_{p=1}^{l} \int_0^{\tau} \Phi(r_{p\varphi}, t) \Big[\beta_i G(r_i - r_{p\varphi}, \tau - t) + \alpha_i \frac{\partial}{\partial n_i} G(r_i - r_{p\varphi}, \tau - t)\Big] dt - \sum_{p=1}^{l} V(r_{pv}) \Big[\beta G(r_i - r_{pv}, \tau) + \alpha_i \frac{\partial}{\partial n_i} G(r_i - r_{pv}, \tau)\Big]. \tag{25}$$

Bearing in mind that the kernels of the integral equations, $L_{ij}(\tau - t)$, depend on the difference $(\tau - t)$, we can reduce the system (14) to a system of ordinary differential equations with constant coefficients or solve it by the Laplace transform. However, from the standpoint of computer implementation of the problem it is more practical to solve the system (14) numerically.

In the matrix equation (14) we replace the variable interval of integration by a finite interval $[0, \tau_m]$ and partition it into m subintervals, not necessarily of equal length:

$$\sum_{k=0}^{m-1} \int_{\tau_k}^{\tau_{k+1}} [L(\tau_m - t)]\{Q(t)\}\, dt = \{H(\tau_m)\}. \tag{26}$$

We invoke the Krylov–Bogolyubov theorem of the mean [3] in order to represent the integral components of Eq. (26) in the form

$$\int_{\tau_k}^{\tau_{k+1}} Q(r_j, t) L_{ij}(\tau_m - t)\, dt = Q(r_j, \xi_k) N_{ij}(\tau_m, \tau_k, \tau_{k+1}), \tag{27}$$

where

$$N_{ij}(\tau_m, \tau_k, \tau_{k+1}) = \int_{\tau_k}^{\tau_{k+1}} L_{ij}(\tau_m - t)\, dt, \tag{28}$$

and $Q(r_j, \xi_k)$ represents the mean value of $Q(r_j, t)$ in the interval $[\tau_k, \tau_{k+1}]$.

The quantity $Q(r_j, \xi_k)$ can be expressed in terms of the values of $Q(r_j, t)$ at the endpoints of the interval:

$$Q(r_j, \xi_k) = \frac{1}{2} [Q(r_j, \tau_k) + Q(r_j, \tau_{k+1})]. \tag{29}$$

Using relations (27)-(29), we can replace the matrix integral equation (14) by the algebraic equation

$$\frac{1}{2}\sum_{k=0}^{m-1}[N(\tau_m,\ \tau_k,\ \tau_{k+1})]\{Q(\tau_k)+Q(\tau_{k+1})\}=\{H(\tau_m)\}. \tag{30}$$

By simple transformations we reduce Eq. (30) to the form

$$\begin{aligned}[N(\tau_m,\ \tau_{m-1},\ \tau_m)]\{Q(\tau_m)\} &= 2\{H(\tau_m)\}-[N(\tau_m,\ \tau_0,\ \tau_1)]\{Q(\tau_0)\}-\\ &-\sum_{k=1}^{m-2}[N(\tau_m,\ \tau_{k-1},\ \tau_k)+N(\tau_m,\ \tau_k,\ \tau_{k+1})]\{Q(\tau_k)\}.\end{aligned} \tag{31}$$

The values of the function $\{Q(\tau_m)\}$ for discrete times are determined by solving the recursive matrix equation (31), in which the values of the vector $\{H(\tau_m)\}$ are given by the relation

$$\begin{aligned}H(r_i,\ \tau_m) &= \gamma\psi(P_i,\ \tau_m)-\frac{1}{2}\sum_{k=0}^{m-1}\sum_{p=1}^{l}[\Phi(r_{p\varphi},\ \tau_k)+\Phi(r_{p\varphi},\ \tau_{k+1})]N_H(\tau_m,\ \tau_k,\ \tau_{k+1})-\\ &-\sum_{p=1}^{l}V(r_{pv})\left[\beta_i G(r_i-r_{pv},\ \tau_m)+\alpha_i\frac{\partial}{\partial n_i}G(r_i-r_{pv},\ \tau)\right],\end{aligned} \tag{32}$$

in which

$$N_H(\tau_m,\ \tau_k,\ \tau_{k+1})=\int_{r_k}^{\tau_{k+1}}\left[\beta_i G(r_i-r_{p\varphi},\ \tau_m-t)+\alpha_i\frac{\partial}{\partial n_i}G(r_i-r_{p\varphi},\ \tau_m-t)\right]dt. \tag{33}$$

The values of $T(r,\ \tau)$ satisfying Eq. (1) and the initial and boundary conditions (2) and (3) are determined by inserting the values of $\{Q(\tau_m)\}$ into relation (7), which by analogy with the foregoing situation we reduce to the form

$$\begin{aligned}\theta(r,\ \tau_m) &= 0.5\sum_{k=0}^{m-1}\sum_{j=1}^{n}[Q(r_j,\ \tau_k)+Q(r_j,\ \tau_{k+1})]N_Q(\tau_m,\ \tau_k,\ \tau_{k+1})+\\ &+0.5\sum_{k=0}^{m-1}\sum_{p=1}^{l}[\varphi(r_{p\varphi},\ \tau_k)+\varphi(r_{p\varphi},\ \tau_{k+1})]N_\varphi(\tau_m,\ \tau_k,\ \tau_{k+1})+\sum_{p=1}^{l}V(r_{pv})G(r-r_{pv},\ \tau_m),\end{aligned} \tag{34}$$

where

$$N_Q(\tau_m,\ \tau_k,\ \tau_{k+1})=\int_{\tau_k}^{\tau_{k+1}}G(r-r_j,\ \tau_m-t)\,dt; \tag{35}$$

$$N_\varphi(\tau_m,\ \tau_k,\ \tau_{k+1})=\int_{\tau_k}^{\tau_{k+1}}G(r-r_{p\varphi},\ \tau_m-t)\,dt. \tag{36}$$

The proposed method has the advantage over finite-difference and finite-element methods that integration over the domain is replaced by integration over the boundary. This feature significantly lowers the order of the resolvents. Also, the system of integral equations has a greater stability of solution than the differential equations.

In the article we have ignored the particular selection of configuration and position for the surface F, as well as estimation of the error of the solution as a function of the order of the system of equations (31); these problems are to be the subject of a special study.

NOTATION

$t,\ \tau$	are the time;
$x,\ y,\ z$	are the coordinates;
r	is the position vector;
$T(r,\ \tau)$, $\theta(r,\ \tau)$	are the functions to be determined;

$\varphi(r, \tau)$, $v(r)$, $\psi(P, \tau)$	are the given functions;
D	is the problem domain;
S	is the surface bounding D;
B	is the domain congruent with D;
Ω	is the surface congruent with S;
F	is the surface exterior to B;
n	is the outward normal to a surface;
P	is the point on the surface S;
a, α, β, γ	are the parameters;
$G(r - r_0, \tau - t)$,	is the Green's function for unbounded space;
$\delta(r - r_0, \tau - t)$	is the Dirac delta function;
$q(r, \tau)$	is the function, nonvanishing only on the surface F.

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LINEAR DEFINING EQUATIONS IN HEAT-CONDUCTION THEORY WITH FINITE THERMAL-PERTURBATION VELOCITY

V. L. Kolpashchikov and A. I. Shnip

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A modification to the general theory of heat conduction with finite thermal-perturbation velocity, in which the linear defining equations are not thermodynamically forbidden is proposed.

In [1] Gurtin and Pipkin developed a general thermodynamic theory of heat conduction assuming propagation of the thermal perturbations at finite velocity. In the framework of this theory, they considered linear defining equations which lead to a linearized heat-conduction equation – in fact, an equation of hyperbolic type. However, the relation between the heat flux itself and the internal energy in this theory is not satisfied by the linear defining equations considered in [1], and therefore the resulting linearized heat-conduction may only be used with great inaccuracy, as a very rough guide.

The present paper outlines a modification of the Gurtin–Pipkin theory such that the linear defining equations (in fact, in terms of new independent variables) are not thermodynamically forbidden.

In the Gurtin–Pipkin theory the defining equations specify at some point **x** and time t the values of the free energy ψ, entropy η, and heat flux **q**, in terms of the temperature at time t, the total history of the temperature $\bar{\vartheta}^t$, and the total history of the temperature gradient $\bar{g}^t$

$$\begin{aligned} \psi &= \hat{\psi}(\vartheta,\ \bar{\vartheta}^t,\ \bar{g}^t), \\ \eta &= \hat{\eta}(\vartheta,\ \bar{\vartheta}^t,\ \bar{g}^t), \\ q &= \hat{q}(\vartheta,\ \bar{\vartheta}^t,\ \bar{g}^t). \end{aligned} \tag{1}$$

The total histories $\bar{\vartheta}^t$ and $\bar{g}^t$ are defined as follows:

A. V. Lykov Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk.

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